Eur. Phys. J. Appl. Phys. **46**, 32608 (2009)
DOI: 10.1051/epjap:2008128

THE EUROPEAN PHYSICAL JOURNAL
APPLIED PHYSICS

Wave reflection by a periodic layered metamaterial

Reflection by a semi-infinite layered structure

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Received: 4 February 2008 / Received in final form: 10 April 2008 / Accepted: 2 June 2008
Published online: 8 August 2008 – © EDP Sciences

Abstract. A technique incorporating the reflection matrix of a semi-infinite discrete layered periodic structure is employed to analyze the electromagnetic properties of a half-space filled with a layered metamaterial.

PACS. 42.25.Fx Diffraction and scattering – 42.70.Qs Photonic bandgap materials – 42.25.Gy Edge and boundary effects; reflection and refraction

1 Introduction

In recent years, research efforts by the microwave and optics communities have been focused on the creation of metamaterials manifesting exotic electromagnetic properties. Generally, metamaterials are constructed as artificial periodic structures. Thus, new theoretical methods are necessary to study electromagnetic waves within complex periodic structures, as well as those reflected by and transmitted through such structures.

In the theory of electromagnetic wave interaction with homogeneous media, the Fresnel formulas [1] fully characterize the fields reflected and refracted by an interface between two media. A similar theoretical description of the reflection and transmission of electromagnetic waves at an interface between artificial, layered periodic media and free space may be useful in metamaterial and photonic periodic structure research as well.

The mode matching between a homogeneous space and a layered periodic structure – as well as between a homogeneous waveguide and a waveguide with periodic structure – plays a crucial role in the application of these artificial materials. The important problem of light confinement by a microcavity in a photonic crystal requires the attenuation of radiation when passing from the microcavity into the cladding [2]. The cause of radiation is the mode-field mismatch at the interface with a semi-infinite periodic structure [3]. The mode-profile mismatch is decreased by introducing one or two tuning sections between the interface and a regular dielectric waveguide [4]. If the reflection from and transmission into a semi-infinite periodic structure were known, the matching problem would be reduced to the design of a tuning layer to be placed at the interface between the regular waveguide and the periodic structure.

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We offer a recently developed theoretical approach [5,6] to describe electromagnetic wave interaction with periodic stacks of metal-dielectric arrays. It is assumed that this artificial metamaterial fills a half-space. Thus, we will deal with electromagnetic wave reflection at an interface with a semi-infinite layered periodic structure, as well as the field penetration into this structure.

First of all, our objective is to offer a numerical method to be used to calculate the generalized reflection matrix of a semi-infinite periodic structure. A key element of the theory we have developed is the use of the periodicity (or shift-symmetry) of the layered structure as a basis to derive a matrix equation to describe the unknown generalized reflection matrix of a semi-infinite structure.

The theory is herein applied to the study of wave reflection by two types of layered structures: singly-periodic planar gratings of metal straight strips, and a semi-infinite stack of biperiodic arrays of C-shaped metal strip elements [7]. We do not apply a limit on the minimum characteristic size of the structure compared to the wavelength of incident light. The resonant case, oblique incidence, and the influence of evanescent waves are all included in the study. However, in this work, we restrict the frequency band such that an array would never produce diffraction orders other than the main ones.

2 Reflection matrix of a semi-infinite periodic structure

Determination of the reflection matrix $\hat{R}$ of a semi-infinite layered periodic structure (see Fig. 1) is most essential element of the theory we have developed. Once a description of wave diffraction at the interface with a semi-infinite structure is known, an inner field as well as a diffraction

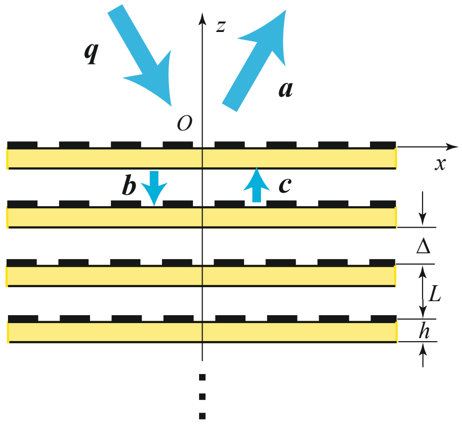

Fig. 1. (Color online) A semi-infinite periodic layered array.

field for the structure of a finite number of layers can be found.

In deriving the reflection matrix $\hat{R}$ of a semi-infinite layered periodic structure, the shift symmetry of such a structure is used. The diffraction properties of a semi-infinite structure will be unchanged if one or any finite number of the layers next to the interface are removed. The equation for $\hat{R}$ can be derived by using this specific symmetry along with the known reflection and transmission matrices of a single isolated layer.

Let us represent the incident, reflected and transmitted fields of a single isolated layer of the layered array under consideration in the form

$$\boldsymbol{E}^i = \sum_{mn} \boldsymbol{q}_{mn} \exp(-i\boldsymbol{\chi}_{mn}\boldsymbol{\rho} + i\gamma_{mn}z) \quad z \geq 0 \tag{1}$$

$$\boldsymbol{E}^r = \sum_{mn} \boldsymbol{a}_{mn} \exp(-i\boldsymbol{\chi}_{mn}\boldsymbol{\rho} - i\gamma_{mn}z) \quad z \geq 0 \tag{2}$$

$$\boldsymbol{E}^t = \sum_{mn} \boldsymbol{b}_{mn} \exp[-i\boldsymbol{\chi}_{mn}\boldsymbol{\rho} + i\gamma_{mn}(z+h)] \quad z \leq -h \tag{3}$$

where $\boldsymbol{\chi}_{mn} = \boldsymbol{e}_x(k_x^i + 2\pi m/d_x) + \boldsymbol{e}_y(k_y^i + 2\pi n/d_y)$, d_x and d_y are the sizes of a rectangular periodic cell, $\boldsymbol{\rho} = \boldsymbol{e}_x x + \boldsymbol{e}_y y$, $\gamma_{mn} = (k^2 - |\boldsymbol{\chi}_{mn}|^2)^{1/2}$, $\mathrm{Im}\gamma_{mn} \leq 0$, and h is the layer thickness. The time dependence is assumed to be in the form $\exp(i\omega t)$.

We describe the incident field as a set of partial plane waves with amplitudes $\boldsymbol{q}_{mn}$, $m, n = 0, \pm 1, \pm 2, \ldots$ If only a single plane wave is incident on the array, the vector $\boldsymbol{q}_{00}$ – corresponding to the main partial wave – is not equal to zero in the set of amplitudes $\boldsymbol{q}_{mn}$. The values k_x^i and k_y^i are components of the wave vector of the main partial wave of the incident field.

The generalized reflection $\hat{r}^+$ and transmission $\hat{t}^+$ matrices of a single layer are determined by the expressions $\boldsymbol{a} = \hat{r}^+\boldsymbol{q}$ and $\boldsymbol{b} = \hat{t}^+\boldsymbol{q}$ for the wave incident on the upper side of the layer. For an incident wave coming from the opposite side, the matrices will be designated as $\hat{r}^-$ and $\hat{t}^-$. The matrices $\hat{r}^\pm$ and $\hat{t}^\pm$ can be found numerically, and they are henceforth assumed to be known.

We define the generalized reflection matrix $\hat{R}$ of a semi-infinite layered structure by the expression

$$\boldsymbol{a} = \hat{R}\boldsymbol{q} \tag{4}$$

where, as above, $\boldsymbol{q}$ and $\boldsymbol{a}$ are the amplitude vectors of the partial waves of the incident and reflected fields. Now, let us introduce two sets of propagating or evanescent partial waves in the gap between the first and the second layers of the structure (see Fig. 1). The partial wave amplitudes of these sets of waves satisfy the equations

$$\boldsymbol{b} = \hat{t}^+\boldsymbol{q} + \hat{r}^-\hat{u}\boldsymbol{c} \tag{5}$$

$$\boldsymbol{c} = \hat{R}\hat{u}\boldsymbol{b} \tag{6}$$

$$\boldsymbol{a} = \hat{r}^+\boldsymbol{q} + \hat{t}^-\hat{u}\boldsymbol{c} \tag{7}$$

where matrix $\hat{u}$ is a propagator matrix that defines the transformation of spatial partial waves on their way through the gap Δ from the surface of some layer to that of the neighbouring one. Equation (6) is a corollary of the above mentioned shift symmetry of a semi-infinite structure. Let us note that the size of the period of a layered structure is $L = h + \Delta$ along the z-axis.

By eliminating vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ from equations (5–7), we arrive at an expression for vector $\boldsymbol{a}$ in the form

$$\boldsymbol{a} = \hat{r}^+\boldsymbol{q} + \hat{t}^-\hat{u}(\hat{I} - \hat{R}\hat{u}\hat{r}^-\hat{u})^{-1}\hat{R}\hat{u}\hat{t}^+\boldsymbol{q}. \tag{8}$$

From the expressions (4) and (8) for vector $\boldsymbol{a}$, a matrix equation may be derived for the unknown matrix $\hat{R}$

$$\hat{R} = \hat{r}^+ + \hat{t}^-\hat{u}(\hat{I} - \hat{R}\hat{u}\hat{r}^-\hat{u})^{-1}\hat{R}\hat{u}\hat{t}^+ \tag{9}$$

where $\hat{I}$ is the identity matrix.

3 The inner field of a semi-infinite structure

With the reflection matrix of a semi-infinite periodic structure known, the amplitudes of the spatial partial waves in the gap between the layers nearest to the interface can be found by using the transfer matrix [8] of a single layer

$$\begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{c} \end{pmatrix} = \hat{T} \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{a} \end{pmatrix} \tag{10}$$

where $\boldsymbol{q}$ and $\boldsymbol{a}$ are the amplitude vectors of the incident and reflected fields, and $\boldsymbol{b}$ and $\boldsymbol{c}$ are the amplitudes of the partial waves propagating within the gap into the structure and towards the first layer, respectively (see Fig. 1). The transfer matrix $\hat{T}$ is expressed in terms of the known reflection and transmission matrices of a single layer

$$\hat{T} = \begin{pmatrix} \hat{t}^+ - \hat{r}^-(\hat{t}^-)^{-1}\hat{r}^+ & \hat{r}^-(\hat{t}^-)^{-1} \\ -(\hat{t}^-)^{-1}\hat{r}^+ & (\hat{t}^-)^{-1} \end{pmatrix}. \tag{11}$$

In this way, the field can be found for any gap within the structure.

Any wave that propagates in a periodic material will be of the Bloch form. The group velocity of such a wave can be found by the expression

$$\boldsymbol{V}_g = \boldsymbol{e}_x \frac{\partial \omega}{\partial K_x} + \boldsymbol{e}_y \frac{\partial \omega}{\partial K_y} + \boldsymbol{e}_z \frac{\partial \omega}{\partial K_z} \qquad (12)$$

where $\boldsymbol{K}$ is the wave vector of the Bloch wave. The averaged energy velocity of the wave is defined as [9]

$$\boldsymbol{V}_e = \frac{\langle \boldsymbol{S} \rangle}{\langle W \rangle} \qquad (13)$$

where the brackets refer to the spatial average within a unit cell of the time-averaged quantities, $\boldsymbol{S}$ is the Poynting vector, and W is the energy density.

4 Numerical results

The equation in $\hat{R}$ is solved numerically. Let us write the matrix equation (9) as

$$f(\hat{R}) = 0 \qquad (14)$$

where

$$f(\hat{R}) = \hat{R} - \hat{r}^{+} - \hat{t}^{-}\hat{u}(\hat{I} - \hat{R}\hat{u}\hat{r}^{-}\hat{u})^{-1}\hat{R}\hat{u}\hat{t}^{+}$$

is the matrix function. The Newton method can be applied to the solution of this equation (14). Following this method, series approximations to the solution are made according to the formula

$$\hat{R}_j = \hat{R}_{j-1} - [f'(\hat{R}_{j-1})]^{-1} f(\hat{R}_{j-1}), \; j = 1, 2, \dots \qquad (15)$$

where $f'(\hat{R})$ is the derivative with respect to the argument of the matrix function, and $\hat{R}_0$ is some initial approximation (e.g., $\hat{R}_0 = \hat{r}^{+}$).

4.1 Reflection by a semi-infinite structure of straight strips

Let us first consider the reflection of a normally incident electromagnetic plane wave by a semi-infinite structure consisting of singly-periodic layers of infinitely long straight metal strips (see insert of Fig. 2). The strips are assumed to be infinitely thin and perfectly conducting. As long as $d_y = \infty$, the summation is reduced to m only in expressions (1–3). Thus, we have arrived at the known, classical problem of electromagnetic wave diffraction by a planar strip grating, the solution of which may be used to calculate the reflection and transmission matrices of a single layer within a layered structure. The rigorous method of analytical regularization of the problem [10] was used in its numerical solution.

The generalized reflection and transmission matrices of a single layer were calculated with the five evanescent partial spatial waves with both positive and negative values of index m taken into account. For normal incidence

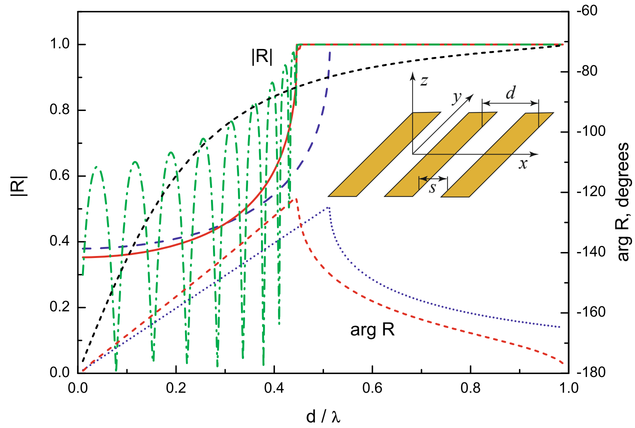

Fig. 2. (Color online) Frequency dependence of the absolute value (solid line) and phase (short-dashed line) of the reflection coefficient of a semi-infinite, layered structure consisting of free standing strip gratings with geometric parameters $s/d = 0.1$, $L/d = 0.3$ for the case of a normally incident plane wave with an x-polarized electric field. As a reference, we also include the absolute value of the reflection from a single isolated grating (dotted line) and a ten-layer structure (dot-dashed line). The frequency dependence of the absolute value and phase of the reflection coefficient when calculated approximately without accounting for any influence of evanescent waves on the coupling between layers are plotted with the dashed and short dotted lines, respectively.

of a plane wave, only the main, zeroth-order partial wave propagates away from the grating plane within the band $0 < d/\lambda < 1$ of the normalized frequency d/λ. Therefore, the size of the generalized matrices is 11×11 within the whole band $0 < d/\lambda < 1$ of the so-called one-wave mode of diffraction. As numerical data show, the number of evanescent waves chosen is quite sufficient to produce accurate numerical results. The difference in reflection and transmission data as calculated with only a single evanescent wave taken into account is very small in comparison with the above mentioned data.

As is well-known, a 2-D periodic structure of infinitely long parallel thin wires is opaque to electromagnetic waves polarized in the direction of the wires for wavelengths much longer than the wire pitch. This structure behaves similarly to a plasma medium for frequencies lower than the plasma frequency [11]. Therefore, it will be interesting to study the interaction of this artificial medium with an orthogonally polarized electromagnetic field. In most cases, such a structure of thin wires is assumed to be fully transparent in the H-polarization. However, we show that reflection at the interface of such a periodic, layered structure may be significant if the strip width is comparable with its period.

Here we show the frequency dependence of the reflection coefficients for interfaces with two semi-infinite structures (each having equal – though different for either of the two structures – distance between layers) consisting of identical gratings (see Figs. 2 and 3).

For both values of the distance between structure layers, there is a transparent band beginning at zero

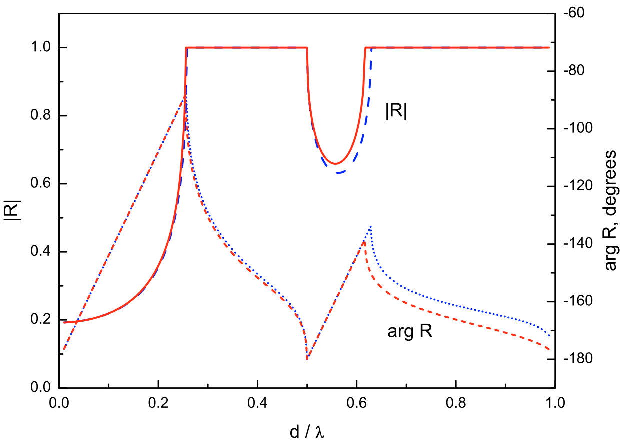

Fig. 3. (Color online) Frequency dependence of the absolute value (solid line) and phase (short-dashed line) of the reflection coefficient of a semi-infinite layered structure of free standing strip gratings with geometric parameters $s/d = 0.1$, $L = d$ for the case of a normally incident plane wave with an x-polarized electric field. The frequency dependence of the absolute value and phase of the reflection coefficient as calculated approximately by neglecting any influence of evanescent waves on coupling between layers are plotted with the dashed and short dotted lines, respectively.

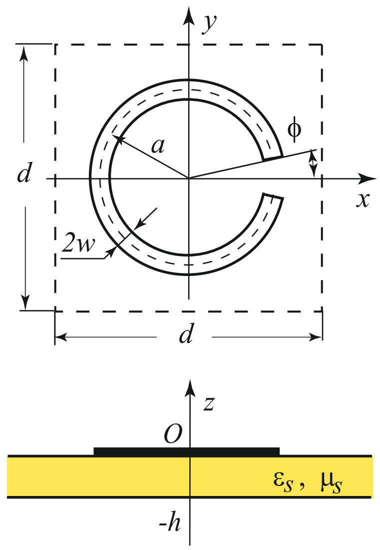

Fig. 4. (Color online) A periodic cell of a strip C-shaped element array: $d = 3$ mm, $a = 1.25$ mm, $\phi = 10°$, $w = 0.05$ mm.

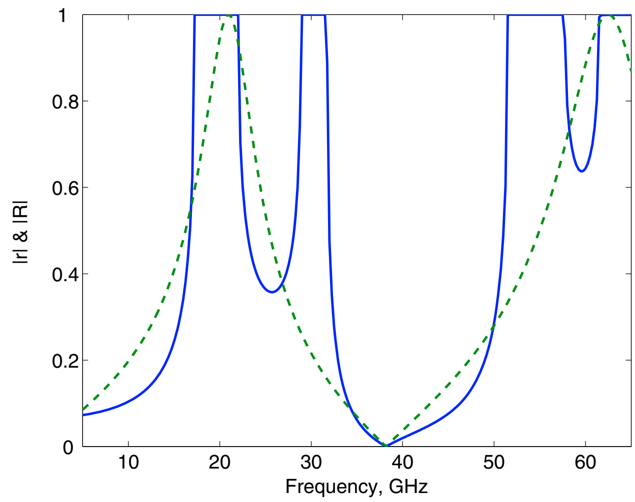

Fig. 5. (Color online) Frequency dependence of the absolute values of the reflection coefficients of a single isolated array of C-shaped elements (dashed line) and a semi-infinite layered array (solid line) for the case of y-polarization and an angle of incidence of 30 degrees.

frequency. At higher frequencies, a forbidden zone appears, covering a wide range of normalized frequencies. These bands of transparency and opacity alternate with frequency. The phase of the reflection coefficient is a nearly linearly increasing function of frequency within the transparent band. A virtual reflecting plane shifts deep into the structure at frequencies within the transparent range, and it shifts back to the interface of the layered medium at the high frequency edge of the forbidden band.

In the case of a structure with closely spaced layers, the evanescent partial waves have a significant influence on the absolute value of the reflection coefficient and on its phase within the transparency band. The edge frequency of the opaque band also essentially depends on the coupling between the layers of the structure due to evanescent waves (see the graphs in Figs. 2 and 3 for a comparison of the values of reflection coefficient calculated exactly and approximately with removal of the effect of coupling between the layers due to evanescent waves). On the other hand, neglecting the contribution of evanescent waves, we arrive at an essential simplification of the mathematical model. In this case, equation (9) can be used to obtain the reflection coefficient instead of the generalized matrix of reflection, and thus the equation is reduced to an ordinary quadric equation that may be solved analytically.

4.2 Reflection by a semi-infinite structure of arrays of C-shaped elements

Let us now consider the oblique incidence of an electromagnetic plane wave on the interface of a metamaterial consisting of identical, biperiodic layers. Each layer is a free standing array of C-shaped perfectly conducting strip elements (see Fig. 4). For the sake of simplicity, we analyze the layered structure of the arrays without substrates. The interlayer distance is $L = 6$ mm.

We consider the case of a y-polarized electromagnetic plane wave obliquely incident in the plane xOz at an angle of $\alpha = 30$ degrees from the interface normal, $k_x^i > 0$. The frequency dependence of the reflection coefficient for the metamaterial interface are shown in Figure 5 and are compared to the reflection coefficient of a single, isolated layer. The properties of a single isolated array of C-shaped elements were simulated using the well-established method of moments [12].

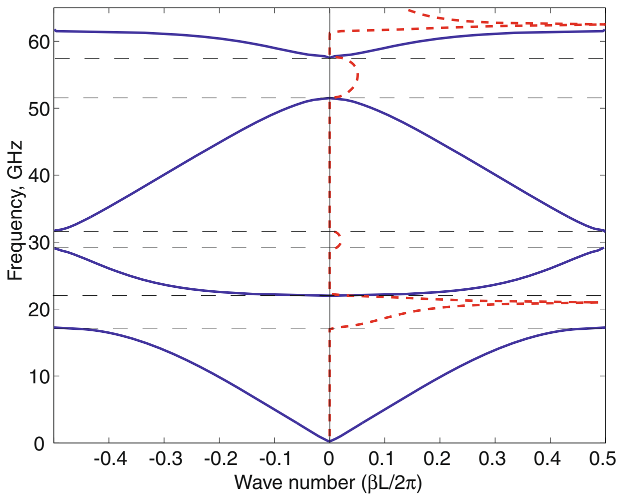

Fig. 6. (Color online) Band structure of the unbounded periodic layered metamaterial of C-shaped planar strips.

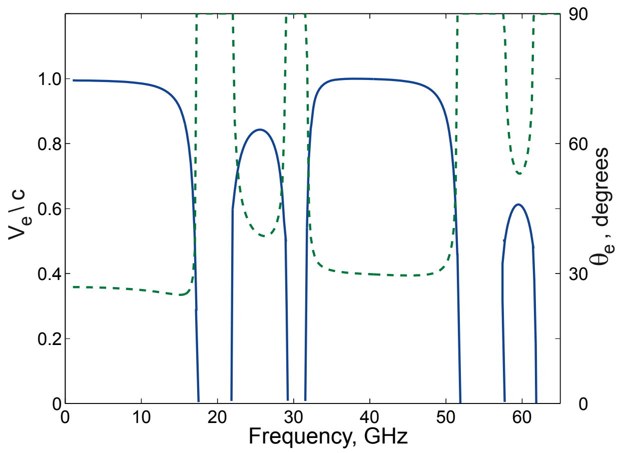

Fig. 7. (Color online) Frequency dependence of the value (solid line) and the direction (dashed line) of the energy velocity of a wave excited in the layered structure. The angle θ_e is defined as the angle between the velocity vector and the negative part of the z-axis.

We restrict our analysis to a frequency band where the isolated array layer produces no diffraction orders other than the main ones. The wave vector of the Bloch wave has two components for the wave incidence case under consideration: $K_x = k_x^i$ and β along the z-axis. Only the component β of the wave vector describes the Bloch wave in this case.

A band structure restricted to the first Brillouin zone of the periodic unbounded metamaterial is shown in Figure 6. The solid and dashed lines show the behaviour of the real and imaginary parts of the z-component of the normalized wave number of the Bloch wave, $\mathrm{Re}(\beta L)/(2\pi)$ and $-\mathrm{Im}(\beta L)/(2\pi)$, respectively.

Forbidden frequency bands may be noted in Figure 6. The imaginary part of the wave number β has a nonzero value only within these frequency bands. The existence of these forbidden bands is due to an interference effect. However, when considering the decrement $\mathrm{Im}(\beta L)/(2\pi)$, a very different dependence can be seen within the frequency band gaps that overlap the frequencies of resonance reflection by an isolated array layer when compared to the band gaps without such an overlap.

The full reflection frequency bands of a semi-infinite layered array correspond exactly to the forbidden frequency bands of the band structure.

It is noteworthy that a backward-travelling wave exists within the third band of the band structure. The band containing the negative slope corresponds to oppositely directed z-components of the group velocity and the phase velocity. As the incident wave is a source of energy, the energy flow is directed into the layered structure from the interface. Therefore, the phase velocity of the backward-travelling Bloch wave is directed towards the metamaterial interface in such a way that its x-component is positive.

The complex amplitudes of the plane waves excited inside the few gaps nearest to the interface of the layered structure were subsequently calculated using expression (10). Phase differences in the wave amplitude only exist for successive gaps between the layers of the structure. The Poynting vector of the field propagating from the interface into the metamaterial was also found. The fields that were calculated satisfy the energy conservation law. We observe an exact fulfillment of the equality

$$S_z = -\frac{1}{2\eta}(1 - |R|^2)\cos\alpha \tag{16}$$

where S_z is the time-averaged z-component of the Poynting vector in the structure and η is the free-space wave impedance.

The energy velocity (13) of a wave propagating in the structure was calculated by using the quantities of the field amplitudes. The frequency dependence of the ratio V_e/c and the direction of energy velocity $\boldsymbol{V}_e$ are shown in Figure 7. We observe a positive refraction over the whole frequency range considered, including the frequency band containing the backward-travelling Bloch wave.

5 Conclusion

A technique incorporating the reflection matrix of a semi-infinite periodic layered structure may be applied to analyze the fields reflected from an interface with a layered periodic artificial material and those excited inside a half space filled with such a structure.

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